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# Identifying Tasks that Promote Creative Thinking in Mathematics: A Tool

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A *tool* constructed to examine a task's potential to stimulate and support creative mathematical thinking was derived from a *hierarchy* of cognitive activities students employ as they solve unfamiliar problems. These cognitive activities informed the formulation of categories of task features that support creative mathematical thinking. Use of this tool is illustrated in a study of the mismatch between potential student responses to a task and actual student responses to the task as implemented. The sufficiency of the tool is discussed.

> "The nation's SET [Science Engineering & Technology] base is held together by *ideas workers…* people who contribute to the creation of new ideas and the effective application of the existing global stock of ideas … Australia must build enterprise and innovation skills by … encouraging more innovative learning and teaching in Australian schools …" (Batterham & Miles, 2000).

To increase the likelihood of Australia sustaining a position on the cutting edge of research design in Science, Engineering, and Technology, school learning environments need to stimulate and nurture creative mathematical thinking. The characterisation of teaching and learning situations that promote creative mathematical thinking would support teacher educators and teachers in pursuit of this goal. The work in progress in this paper describes the construction of a tool to analyse tasks for the purpose of evaluating their potential to stimulate creativity. Contrast of the potential and actual student responses to a task found to trigger some creative thinking provided a way to study the mismatch and identify characteristics of the task as implemented that influenced the stimulation of creative thinking.

## Literature Review and Construction of Theoretical Framework

Krutetskii (1976) described the mathematically insightful behaviour (mental activities) of highly capable students as they thought aloud to solve unfamiliar challenging problems. I used Krutetskii's empirical data to construct a hierarchy of Krutetskii's mental activities (Williams, 2000) and noted Krutetskii's language for describing these (mental) cognitive activities was consistent with the language used in Bloom's (1956) Taxonomy of Learning Objectives that has since been adapted to describe categories of higher-level thinking in educational settings (e.g., Tannenbaum, 1983). The cognitive activities in my hierarchy were: (a) repeat taught information (Recall); (b) understand a learnt concept (Comprehension); (c) apply a previously learnt procedure (Apply); (d) build on a previously known idea to solve a problem with a slight twist; recognise the need for extra information (Analysis); (e) work backwards; use more than one pathway; explain the need for extra information when insufficient information is provided to solve a problem (Analytic-synthetic); (f) combine concepts to create an original concept (Synthesis); and (g) recognise inconsistent information (Synthesising-evaluation, a subset of Evaluation).

Dreyfus, Hershkowitz and Schwarz (2001) have used three observable nested elements to study the process of abstraction (*Recognising*, *Building-with* and *Constructing*).

Recognising and Building-with are nested within Constructing. These nested elements of abstraction can be described in terms of the categories in my hierarchy. Recognising includes the process of identifying a context in which a previously abstracted entity applies (Comprehending) and recognising a previously abstracted entity can be applied to a given context (Applying). Building-with includes using previously abstracted entities as part of several different processes such as: applying a previously abstracted entity in a known context (Applying), or in a new context (Analysing), or applying several previously abstracted entities in a familiar order (Applying) or a different order (Analysing). The relationship between the observable elements of abstraction and my categories of cognitive activities suggests my term "hierarchy" should be replaced by *nested categories of increasingly intellectually complex activity*. Constructing—mathematically insightful behaviour—is a process of integrating previously abstracted entities to develop a new mathematical insight (Synthesising & Evaluating). Constructing may occur in a more expert-directed learning culture where suggestions, hints, and corrections are made or constructing may be a more autonomous activity where hints are not provided but questions are asked for the purpose of eliciting more complex thinking (Williams, 2000).

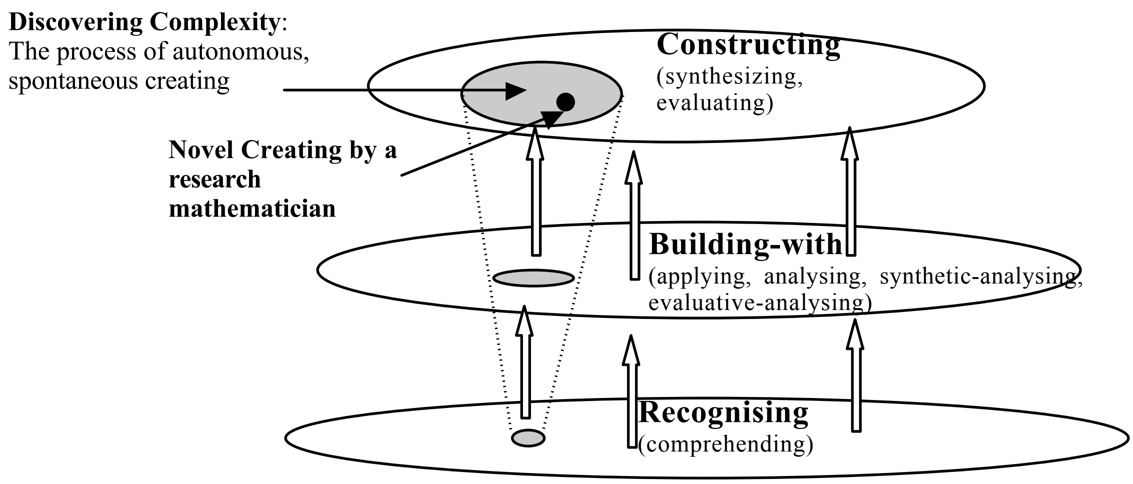


Figure 1. Relationship between observable nested elements in the process of abstraction, the process of discovering complexity and the process of employing cognitive activities.

Chick (1998) described her cognitive processes as a research mathematician "not only choosing the cues and concepts—and often unexpected cues and concepts—but even the very question" [pp.17]. Senior secondary mathematics students discovering complexity and spontaneously formulating their own questions exhibited characteristics of *flow* as they worked with unfamiliar mathematical ideas to resolve their questions (Williams, 2000). Csikszentmihalyi (1992) described flow as intense engagement where people lose all sense of self, time, and the world around. He found necessary conditions for flow included working autonomously just above the persons present skill level on a challenge almost out of reach. Figure 1 above and Table 1 below relate Dreyfus, Hershkowitz and Schwarz's (2001) elements of abstraction, my categories of cognitive activities, the process of discovering complexity (Williams, 2000) and Chick's (1998) cognition as a mathematician.

Table 1

Nested Cognitive Processes Associated with Mathematically Creative Behaviour

| *Cognitive Category* | *Example of cognitive activity elicited and symbol for research source from which the activity was derived* |
|---|---|
| Evaluating. (Constructing) | * + Progressively reflect on the situation as a whole for the purpose of recognising inconsistent information and/or finding a more elegant solution pathway.<br>+ Reflect upon the process of problem solution for the purpose of recognising its limitations and applications to other contexts.<br>^ Reflect upon the solution pathway developed and its possible contribution to generic mathematical processes for future use. |
| Synthesising. (Constructing) | * + Explore the problem from many perspectives rather than just work towards a solution.<br>+ Formulate mathematical arguments to explain discovered patterns.<br>+ < Progressively explore the problem to continually develop new insights and simultaneously consolidated “fragile” new insights.<br>* + Integrate concepts to create original concepts. Could vary in:<br>• + Number of concepts involved<br>• + # Diversity of the domains concepts were drawn from<br>• + Size of the conceptual leap<br>• + Spontaneity of the question pursued. |
| Evaluative-analysing (Consolidating) | + Interconnect solution pathways for the purpose of identifying flaws and strengthening arguments.<br>+ Find fast ways to assess the reasonableness of a result. |
| Synthetic-analysing (Building-with, Consolidating) | * + Use more than one pathway to solve a problem.<br>+ Interconnect various representations, operations, and assumptions through comparison of different pathways.<br>* + Explain why extra information is needed when insufficient information is provided to solve a problem.<br>* + Work backwards to find what is usually an initial condition when given the answer and the other initial conditions.<br>^ Recognise and utilise previously known mathematical ideas in a new sequence to explore a situation. |
| Analysing (Building-with, Consolidating) | * Solve using a previously known process with a slight twist.<br>+ Familiarise self with problem using specific numerical examples.<br>+ Systematise the numerical results and search for patterns.<br>* + Recognise insufficient mathematical information has been provided.<br>^ Apply previously known mathematical procedures in a new context. |

Key: * Krutetskii (1976); # Chick (1998); + Williams (2000);
^ Dreyfus, Hershkowitz, & Schwarz (2001); < Dreyfus & Tsamir (2001)

My collaborative groups of senior secondary mathematics students met Csikszentmihalyi's (1992) conditions for flow because they continually posed and resolved their self-set questions involving intellectual challenges (mathematical intellectual complexity) by using mathematical ideas above their present skill and concept level (increased conceptual complexity) (Williams, 2000). Table 1 lists categories of cognitive activity in decreasing order of intellectual complexity in Column 1. Column 2 contains examples of student responses used to infer cognitive activity. These examples are drawn from research into students' thinking as they solved unfamiliar challenging problems (Krutetskii, 1976; Dreyfus, Hershkowitz, & Schwarz, 2001; Williams, 2000; Chick, 1998; Dreyfus & Tsamir, 2001). Evaluative-analysing is a new category associated with examining the reasonableness of findings as opposed to reflecting on the task overall (Evaluating). This category was empirically developed from responses of students in my classrooms and a similar category was proposed Anderson's in revisions to Bloom's Taxonomy (1998).

Table 2

Task Features that Promote Autonomous, Spontaneous, and Creative Thinking

| *Nested Student Responses* | *Task Features* |
|---|---|
| Autonomous Access. | Variety of representations and/or initial assumptions.<br>Work with specific cases or generalisations.<br>Initial specific cases leading to later general arguments. |
| Spontaneous Pursuit. | Overarching task question sufficiently unusual to be unfamiliar to all.<br>Over-arching question contains an *unusual twist*.<br>Potential for idiosyncratic discovered complexity.<br>Potential for successive discovered complexities (sustained challenge).<br>Apparent inconsistencies or generic questions to elicit complex thinking. |
| Creative Execution. | Conceptual, general, or multiple solutions rather than an exact numerical solution provides scope for:<br>Unusual solution pathways<br>Non-linear progression<br>Pursuit of elegance<br>Simple elegant checks for inconsistencies<br>Assembly and integration of unexpected mathematical ideas. |

Table 2 (Column 1) displays nested categories of student responses associated with creative mathematical thinking (Autonomous Access, Spontaneous Pursuit, and Creative Execution). For example, autonomous access is nested within spontaneous pursuit. Task features relevant to each nested category of student response were developed by considering the cognitive activities described in Table 1 and identifying task features that support these types of activities (Table 2, Column 2). *Autonomy* for the purpose of this study refers to student choice in selecting: the representations to explore; the assumptions

they apply; and to enter the task is through specific examples or general arguments. *Access* refers to the students' opportunity to find an entry point to the task and student opportunity for sustained exploration of relevant mathematical ideas. *Spontaneity* refers to student-focused inquiry triggered by curiosity. *Pursuit* refers to student activity related to resolving their self-focused questions. A high degree of *creativity* exists where students spontaneously formulate highly relevant questions and assemble and integrate mathematical ideas from mathematical topics—which until now appeared to be unconnected—for the purpose of resolving these questions using elegant solution pathways. *Execution* refers to the quality of the creativity and is relative to (a) the prior mathematical background of the student; (b) the recentness of exposure to the mathematical ideas upon which they drew; (c) the students' prior knowledge of connections between the mathematical ideas they integrated, and (d) the elegance of the argument produced.

## Rationale, Methodology, and Research Question

This research was undertaken in a Year 8 mathematics classroom in a large metropolitan city in Australia. The particular lesson and task studied were selected from a data set of fourteen consecutive lessons from a classroom that was part of the broader international Learners' Perspective Study of teachers seen to "display good teaching practice". The methodology for the broader study and a more detailed account of the lesson in which this task was situated has been provided in a previous paper (Williams, 2001) The classroom research undertaken used three video cameras to simultaneously capture: the whole class, the teacher, and two focus students. The two focus students took part in individual video-stimulated reconstructive interviews. Evidence was drawn from the video data, classroom field notes, and student and teacher interview data. The research questions discussed here are "Is this tool sufficient to account for the actual student responses to the task as implemented?" And "Does the contrast of potential and actual student responses inform my study of factors that promote or inhibit creative student thinking?"

The task was presented to the students in Lesson 3 after the topic of perimeters of rectangles. This task was selected as the illustrative example for this report because it contained many of the task features identified to promote creative thinking (Table 2) but during task implementation creative mathematical thinking was not evidenced. After task closure, two students (Leon and Pepe) displayed creative thinking in further exploration of the task. As Leon, and Pepe were not the focus students, classroom dialogue was captured in fragments by the video cameras and in the field notes. Leon's interview (Lesson 12) supplemented this data.

Two distinct learning cultures operated at different times in this classroom. One culture was based on teacher selection of direction and student contribution of ideas whereas the other culture was based on student selection of direction and teacher provision of assistance. In whole class discussions, the teacher selected the solution pathway and questioned students (video data) for the purpose of building on their ideas (teacher interview). The teacher decided which student comments to integrate into whole class discussion and acknowledged then discarded other comments. In small group work (usually pairs), students selected their own solution pathways and consulted the teacher when they experienced difficulties. The teacher generally provided hints or explanations and sometimes asked open questions. The task: ‘What are the dimensions of the rectangle with perimeter 38 cm?’ was undertaken as a whole class activity.

## Analysis and Results

Table 3 contrasts the potential and actual student responses to the task. The task has the *potential* for exploration of unfamiliar mathematical ideas; the perimeter was given and working backwards was required to find the dimensions. As only one of three unknowns was given, substitution and algebraic manipulation would not provide a numerical solution. The existence of more than one rectangle with a perimeter of 38cm provided an ‘unusual twist’. Some examples of the potential discovered complexities include (a) How does the pattern L + W = 19 relate to the perimeter formula or the context? (b) Why are the values of L and W restricted? (c) What mathematical arguments explain what happens? The conceptual ideas accessed may commence with generalisations or with tabulation of specific examples to find a pattern and then explain why it exists. These complexities involve student connection of solution pathways using various representations (specific numerical examples, tables of numbers, numerical patterns, algebraic patterns, algebraic variation to provide a general argument). Different assumptions are possible (number of decimal places) and elegant solutions may result (argument through variation using algebra).

In the *actual* task as implemented, the mixed ability of the students led to the teacher’s decision to work at a pace that catered adequately for slow pace students but did not extend the ‘brighter’ students to their full potential (teacher interview). Most students (including ‘brighter’ students) reported how much they enjoyed being in this teacher’s class and how much more they learnt than in most mathematics classes. The teacher selected a pathway that progressed from specific examples to tabulation to discussion of a student-recognised pattern to algebraic manipulation to confirm the pattern. The teacher integrated Susan’s discovery of a pattern into the whole class discussion. She acknowledged the correctness Leon’s recognition of the complimentary variation of dimensions to retain the same perimeter but she did not integrate this into the discussion. Leon’s explained his use of the dimensions [9, 10] to find the width when the length was 9.6 by “moving the point six around the corner”. This potentially could have been a precursor to a mathematical argument based on variation within the physical context.

After teacher questioning that explored the number of possibilities using one decimal place, the task was brought to closure with the question “How many possibilities if there were two decimal places?” (Posed but not answered). During the next few minutes (at the commencement of pair-work), Leon and Pepe exhibited an intensity of focus as they talked together at a fast pace and projected questions across the room to the teacher. This episode is described in more detail in a previous paper (Williams, 2001). Leon’s interview suggests Leon and Pepe discovered a complexity focused around: What is the relationship between the number of rectangles possible using dimensions correct to different numbers of decimal places? Leon’s interview descriptions provided evidence of creative thinking.

> ‘… it kept building and building and building- you sort of found a formula- a way to categorise it’

Although there is insufficient evidence to determine whether Leon and Pepe resolved their spontaneous question, they did assembled disparate mathematical ideas and attempt to formulate an algebraic expression to show the relationship between the terms (synthesising).

Table 3

Contrast of Potential Student Response and Actual Student Response

| *Student responses* | *Potential task features (examples are illustrative not exhaustive)* | *Actual task as implemented* |
|---|---|---|
| Autonomous Access.<br><br>Pathway choices | 1) Representations: specific numbers, tabular, physical context, algebra.<br>2) Assumptions: integers, number of decimals places, all integers and decimals.<br>3) Conceptual complexity of pathway (specific dimensions, physical variation in context or general variation of formula).<br>4) Specific progressing to general through generation of specific dimensions, finding a pattern, explaining why this pattern exists. | No autonomy of pathway (teacher selected). Some autonomy in nature of student comments contributed to the class discussion. No autonomy in selection of student comments to explore further. |
| Spontaneous Pursuit.<br><br>Idiosyncratic foci | 1) Work in opposite direction but insufficient information to solve by substitution.<br>2)'Unusual twist': more than one solution.<br>3) Discover complexity: For example, how does the specific link with the physical and the algebraic? How can findings be checked or what mathematical arguments exist?<br>4) Successive discovery of complexity: progressive links between various complexities.<br>5) Task triggers to more complex thinking: Why was more than one rectangle possible? | The nested nature of autonomy within spontaneity accounts for lack of evidence of spontaneous questions. Several insightful comments (not questions) occurred:<br>[0,19] will be a straight line; and<br>Use of [9, 10] to get 9.4. |
| Creative Execution.<br><br>Questions asked, strategies and ideas assembled, novelty and elegance of composition | 1) Non-linear exploration: evidenced through discovered complexities previously discussed.<br>2) No exact numerical solution: existence of many possible rectangles and general arguments about why this is possible.<br>2) Unusual solutions: e.g., Developing an algebraic series from solutions to different numbers of decimal places.<br>3) Elegance (General arguments using variation in the physical context and/or algebra).<br>4) Fast check (connection of variation argument in context and algebra).<br>5) Unexpected concepts: restrictions linking algebra and context; arguments using variation. | Leon and Pepe worked autonomously after task closure to discover complexities encapsulated by the spontaneous questions: 'Why does the number of answers keep building and building?' 'How are they related? 'Can we use a formula to show the relationship?' They selected their own foci, and created an elegant and unusual pathway—algebra to relate 'answers'. |

## Discussion and Conclusions

This analysis tool has the potential to increase teacher awareness of the complex cognitive activities associated with creative student thinking and inform their pedagogical decisions about task selection and implementation. The sufficiency of this tool in accounting for student response to this task was demonstrated. The tool's usefulness in identifying the mismatch between potential and actual student responses was confirmed and has assisted in identifying factors that promoted and inhibited creative thinking. The provision of a task with the potential for creative thinking was not sufficient to trigger such thinking in the absence of student autonomy. The effect of the classroom culture on promotion of creative thinking was clearly demonstrated through the change of classroom culture during this lesson; the autonomy to explore aspects of this task after task closure was a significant aspect of the changed classroom culture that supported creative thinking. Areas for further study include testing the tool's sufficiency with a diverse range of tasks demonstrated to elicit creative thinking, and exploring triggers that may elicit increasingly more complex thinking.

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