A LONGITUDINAL STUDY OF CHILDREN'S THINKING ABOUT DECIMALS: A PRELIMINARY ANALYSIS

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Abstract

This paper reports preliminary results of a longitudinal study of children's understanding of decimal notation. A large sample of students completed a short test that enabled their understanding to be classified into four categories and changes over periods of up to two years to be tracked. When they attained expertise, students almost always retained it, even if it is simply following memorised rules. A small core of students retains "longer-is-larger" misconceptions. In contrast, students seem to move in and out of "shorter-is-larger" misconceptions. Improvements and hypotheses to be investigated in the future are noted.

Introduction and Background

It is now well documented that many students throughout schooling and indeed many adults have difficulty understanding the notation used for decimal fractions. The recent Third International Mathematics and Science Study showed that internationally about a half of 13 year old students could select the smallest decimal number from a multiple choice list of five decimals (data held at the Australian Council for Educational Research). Similar results have been known for many years in several countries. The aim of this paper is to present preliminary results from a longitudinal study that is tracing the development of students' thinking about decimal notation. This is of interest in its own right because being able to interpret decimals is important in a variety of everyday contexts (e.g. interpreting digital displays and calculator answers) as well as for mathematical tasks such as rounding and using significant figures. It is also a case study of how students' understanding and misunderstanding develops with progress through school and in the context of various types of instruction.

There are several ways of classifying the erroneous rules that students may apply when ordering decimals (Resnick, Nesher, Leonard, Magone, Omanson & Peled, 1989; Sackur-Grisvard & Leonard, 1985). The coarsest classification is that some students select "longer is larger" (e.g., deciding 0.125 is larger than 0.3) whilst others select "shorter is larger" (e.g., deciding 0.3 is larger than 0.496). Stacey and Steinle (1998), working with interview and written data, traced the various ideas behind these erroneous rules, identified further misconceptions and developed a diagnostic test. This Decimal Comparison Test takes about five minutes and asks students to select the larger from 30 pairs of carefully chosen decimals. It enables ten patterns of thinking to be diagnosed. Some of these patterns of thinking are "longer-is-larger" misconceptions, some are "shorter-is-larger" misconceptions and others belong to neither of these. Steinle and Stacey (1998) present evidence that some misconceptions about decimal notation appear to be the result of instruction. In other cases, these misconceptions arise when ideas interfere with each other. Although future analyses will use the refined classifications, this paper reports student progress only in terms of four major categories:

- longer-is-larger misconceptions (resulting from any of five identified patterns of thinking and possibly others),
- shorter-is-larger misconceptions (resulting from three identified patterns of thinking and possibly others),
- apparent-experts (may possess excellent understanding or may apply correct rules not understood or may have an identified misconception (Steinle et al (1998))
- unclassified (since the criteria for classification are quite stringent, this large group includes students thinking about decimals in unknown ways and others who are inconsistent).

Cross-sectional data (see Figure 1, taken from Stacey and Steinle, 1998b) provides a picture of the incidence of various ways of thinking about decimal notation and how it varies with age. The Longer-is-larger category decreases from Grade 5 (32%) to Year 10 (5%), the trend suggesting that it is unlikely to be common in adult life. The Shorter-is-larger category plateaus at about 10%, which suggests that this general belief may continue into adulthood. The percentage of task experts also plateaus to about 60% in Year 10, which suggests that there are many adults who have difficulty understanding decimal notation. The task expert category of Figure 1 is somewhat smaller than the apparent-expert category used in this paper, because students with identified misconceptions have been removed from it and placed with unclassified students to form the category "Other". This adjustment is of the order of 5% and so Figure 1 can be taken as a reasonable guide to the number of students in the four categories used in this paper. This paper moves from the cross-sectional analysis to the beginning of a longitudinal analysis, which traces the movement of individuals in the overall data and reports on two questions:

- do students stay in the same category or move frequently from one to the other?
- what are the common paths through the misconceptions to attaining expertise?

The longitudinal sample and testing

This section presents preliminary results of the longitudinal study from 1995 to 1997. The sample was originally selected to contain a good mix of schools and to maximise the possibility of following students when they changed from primary (Years 0 to 6) to secondary school (Years 7 to 12) at approximate age 12 years. It consists of classes from:

- one state secondary school in a low socio-economic area and its three "feeder" primary schools,
- one church secondary school in an middle socio-economic area and its main feeder primary school,
- one private girls school in a high socio-economic area with both primary and secondary students,
- two large state primary schools situated in the same middle socio-economic area and the two high schools to which their students mainly progress and
- one church girls' secondary school in a high-middle socio-economic area.

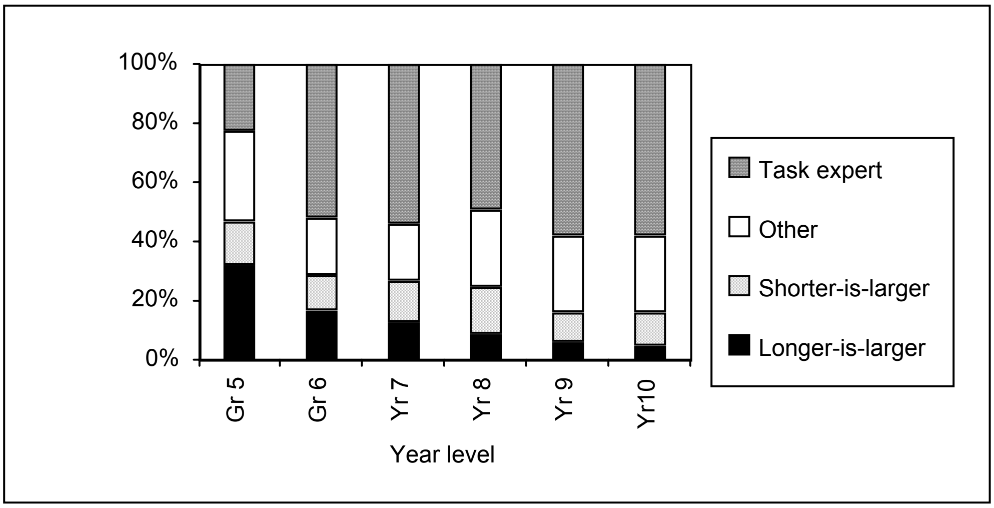

Figure 1: Distribution of classification by grade/year level

Students were tested with the Decimal Comparison Test at most once every six months, making a total of five testing times in the data under consideration, from the end of 1995 to the end of 1997. However, schools tested less often than this, for various reasons including different dates for joining the program. In this period, no individual student or class completed the test more than four times. The year level distribution of students is shown in Table 1. Note that many students are counted more than once, some up to four times each. In total, 5383 tests have been analysed, although there is no longitudinal data yet for many students. The large numbers in the lower year levels is due to selecting students whose progress could be followed until 1999.

Table 1. Year level distribution of students completing test 1995-1997

Year level	Year 4	Year 5	Year 6	Year 7	Year 8	Year 9	Year 10	Total
Number of students	336	965	874	1690	658	497	363	5383

The number of students who have completed the Decimal Comparison Test exactly one, two, three and four times is shown in Table 2. The students who have completed only one test play no further role in the analysis in this paper, as it aims to track change of individuals from one test to others. For this analysis, the first time an individual undertook the test will be called Test 1, the second time will be called Test

2 and so on. The tests are numbered for the individual, rather than by the date administered. Therefore for some students Test 1 was in 1995 while for others it was in 1996 or 1997. For some students, Tests 1 and 2 have been taken six months apart, whereas for others they may be one year or even 18 months apart if the student was absent on some testing days. This is an unsatisfactory feature of this preliminary analysis that will be addressed in subsequent work.

Table 2. Number of students by number of tests completed.

Number of tests	One test	Two tests	Three tests	Four tests
Number of students	1590	1198	307	119

Results

Changes of classification over consecutive tests.

Table 3 illustrates the changes in classification that occur over consecutive tests. The abbreviations A, L, S and U refer to apparent-expert, longer-is-larger, shorter-is-larger and unclassified groups, respectively. The cross-tabulation shows the movement of students who have tested in one category to other categories at the individual's next testing. The numbers are amalgamated from all tests. The data is therefore from Test 1 followed by Test 2, from Test 2 followed by Test 3 and from Test 3 followed by Test 4. As noted above, these changes are mostly over a period of about six months, but will also include changes over longer periods where students missed out on intermediate testing. This anomaly in the data will be eliminated when the final analysis is done, to give a better measure of change over six months.

The 426 students who have done the test more than twice contribute several times to the data. To illustrate, there were 165 instances where a student showed longer-is-larger thinking at one test and was an apparent-expert when next tested. Some of these individual students will have been tested again and contribute to the table again, in the group of 835 students who are apparent-experts on the prior classification.

Table 3. Changes in classification over consecutive tests (N = 2169)

Earlier classification	**Later classification**			
	A	**L**	**S**	**U**
A (N= 835)	732 (88%)	15 (2%)	29 (3%)	59 (7%)
L (N= 732)	165 (33%)	334 (46%)	94 (13%)	139 (19%)
S (N = 348)	125 (36%)	44 (13%)	120 (34%)	59 (17%)
U (N = 254)	98 (39%)	49 (19%)	50 (20%)	57 (22%)

Table 3 shows that from one test to the next, almost all of the apparent-experts stayed as apparent-experts and about one third of other students became apparent-experts. Nearly half of the longer-is-larger students (in fact two-thirds of those who did not become experts) re-tested as longer-is-larger. The shorter-is-larger students moved more than the longer-is-larger, but still about one third stayed in the same category.

Amongst those who did not become experts, over half remained as shorter-is-larger. As might be expected, the unclassified students spread most evenly across the other misconception categories. It will be important to repeat this analysis separating students by age group, as age is likely to be an important determinant of the speed and direction of change.

Changes of classification over at least one year.

Table 4 is similar to Table 3 except that it shows the changes in classification that occurred between testings typically at least twelve months apart. The numbers are amalgamated from Test 1 to Test 3 and Test 2 to Test 4. Therefore individuals who have been tested four times (there are 119 of these) contribute twice to the 545 comparisons in the table. The comparisons in this table come from students across a wide age range. About two thirds of the sample were in Year 4 or 5 at first testing. The oldest was a small group tested once per year in Year 8, 9 and 10. Some of the data is over one year, from students who were tested regularly, but where tests were missed it is over a period of up to 2 years.

Table 4 Changes in classification over at least one year (N = 545)

Earlier classification	**Later Classification**			
	A	**L**	**S**	**U**
A (N= 135)	125 (93%)	2 (1%)	1 (1%)	7 (5%)
L (N= 271)	104 (38%)	94 (35%)	23 (8%)	50 (18%)
S (N = 80)	44 (55%)	9 (11%)	19 (24%)	8 (10%)
U (N = 59)	25 (42%)	14 (24%)	8 (14%)	12 (20%)

Table 4 again shows that almost all of the apparent-experts re-tested as apparent-experts whilst about two-fifths of longer-is-larger students and unclassified students and over a half of the shorter-is-larger students became apparent-experts. This is consistent with shorter-is-larger thinking being somewhat more sophisticated than longer-is-larger thinking. Of those classified students who do not move to being experts, most stayed in their original category (56% of longer-is-larger and 52% of shorter-is-larger) but there was some movement into other misconception categories, not just towards expertise. The unclassified students spread most evenly across the other misconception categories.

We had previously expected that it would be more likely that longer-is-larger students would become shorter-is-larger than vice versa, because we hypothesised that students may move into the more sophisticated misconception on the way to expertise. However, since movement is equally frequent in both directions in both Tables 3 and 4, the data does not support this hypothesis. A further analysis using the finer grained classification system and broken down by age is required.

Changes of classification over two years.

Table 5 is similar to the previous two tables except that it shows the changes in classification that occurred from Test 1 to Test 4. In this sample, the only students who had been tested four times were from the two large primary schools in the middle socio-economic area. They were all tested initially in Year 4 and for the fourth time in the second half of Year 6 or in Grade 5 initially and for the fourth time in the second half of Year 7, when they had moved to a nearly secondary school. For this reason, the Table is labelled as indicating changes over two years.

Table 5. Changes in classification over two years (N = 119).

Earlier Classification	**Later Classification**			
	A	**L**	**S**	**U**
A (N= 16)	16 (100%)	0 (0%)	0 (0%)	0 (0%)
L (N= 74)	46 (62%)	10 (14%)	5 (7%)	13 (18%)
S (N = 11)	9 (82%)	2 (18%)	0 (0%)	0 (0%)
U (N = 18)	15 (83%)	0 (0%)	2 (11%)	1 (6%)

Table 5 again shows that all the students who were apparent-experts at first testing stayed as apparent-experts. These students are likely to have been among the most able students (having achieved early expertise in Year 4 or Year 5) so it is not surprising that they test again as experts at the fourth testing. Most of the shorter-is-larger and unclassified students became apparent-experts over the two years, which again supports the observation that these students have more sophisticated thinking than the longer-is-larger thinkers. In the latter group, only about two in three achieved apparent-expert status. At this level of analysis, there is no evidence to support the hypothesis that students who tested initially in one category, but do not move to expertise, stay over two years in the same category. However a closer look at the initially longer-is-larger students who move to the unclassified category is warranted. Because of the strict definitions employed for categorisation, it is possible that some of the 18% of formerly longer-is-larger students may still hold broadly longer-is-larger ideas yet not meet the criteria for that classification. This analysis would give a better insight into whether these students are essentially stuck in the one category or are on the way to expertise.

How representative is this group of students who have done the test four times? There are several reasons why they may be a better group than the rest of the sample. To be present on the four days of testing indicates that are likely to be regular school attenders with relatively stable schooling. The fact that their teachers have made the effort to test four times indicates commitment on their part. By Test 4 in Years 6 or 7, there are 86 (72%) apparent-experts and only 12 (10%) longer-is-larger thinkers. These proportions do indeed seem somewhat better than the proportions reported previously for the whole sample and summarised in Figure 1 above. This bias will require careful treatment in the next analysis.

A first attempt at following individuals

Table 6 is an initial view of the paths that students take through a series of tests. It displays data from the 119 students who completed the test four times (the same data set as Table 5). As noted above, these students were either in Year 4, moving through to Year 6 or in Year 5 moving through to Year 7. Each cell shows the number of students in the category at that particular test. There are, for example, 86 students (72% of the total of 119) testing as apparent-experts at Test 4. At Test 1, 16 students tested as apparent-experts. The table also records that all 16 of these were apparent-experts at Test 4. Similarly there were 34 apparent-experts at Test 2 and 32 of these individual students were apparent-experts at Test 4. Combining the information in Table 6 with the data in Tables 3, 4 and 5, we see that there is very little movement out of the apparent-expert category, suggesting that students retain knowledge of how to complete the test.

Table 6. Numbers of individuals in each category at each of the four testing times and, in brackets, the number of those students who retested in the same category at Test 4.

	Test 1	Test 2	Test 3	Test 4
Apparent-expert	16 (16)	34 (32)	48 (45)	86
Longer-is larger	74 (10)	48 (6)	33 (8)	12
Shorter-is-larger	11 (0)	23 (4)	19 (2)	7
Unclassified	18 (1)	14 (2)	19 (7)	14

Most of the students who have tested as longer-is-larger initially have also become experts by Test 4. The number in this category steadily reduces. However, Table 6 indicates that there is probably a small core of students who remain persistently in the longer-is-larger category. This will be further investigated with the finer grained analysis and is expected to be a larger effect in a less selective sample.

The shorter-is-larger category shows a different pattern. This category grew in Tests 2 and 3, but the low numbers who persisted in this category in Test 4 indicate that students move in and out of it over time. This is consistent with our experiences when we interviewed students from a class of Grade 5 and 6 students (not in this sample). These students had recently been studying fractions and so it seemed that the number of shorter-is-larger students who interpreted decimals as fractions (reciprocal thinkers) was inflated by the recent experience. The pattern of pathways for students who are unclassified seems similar to the shorter-is-larger pattern.

Discussion

The purpose of this paper was to report a preliminary analysis of data showing students' progress in their understanding of decimal notation. Several ways in which the analysis can be made more revealing have been highlighted: by following individuals, by following classes so that teaching effects can be observed, by separating the analysis by age group and by using the finer classification system. This will be done when an extra year has been added to the data set, so that there are more long runs of data from individuals and more histories of intact classes.

The preliminary analysis has provided the following results to be confirmed later. There is quite marked stability of classification. Of course there is a general trend towards expertise, but those students who do not achieve expertise tend to remain in the same category. Even after a passage of at least a year (i.e. first to third test) about half of the students retaining a misconception are classified in the same way. This is a significant result given the stringency of the classification criteria and it confirms our informal data (from interviewing previously classified students) that the test, although taking only a few minutes, is highly reliable.

Students in different classifications behave differently. Apparent-experts nearly always stay in this category. This would be expected of students who "really understand" decimals. However, at least in the context of this test, the skill of decimal comparison is well retained even by those who use a rote-learned rule (e.g. compare digits from left to right or add zeros).

A small group of students seem to persist in the longer-is-larger category. On the other hand students seem to move in and out of the shorter-is-larger category and are more likely to move to expertise. Unclassified students are similarly likely to move towards expertise, contrary to a previous finding on a group of 50 students (Moloney and Stacey, 1997) they were more likely to move to expertise than the shorter-is-larger students. Following individual paths will help in unravelling students' thinking further and eventually providing better guidelines for teachers.

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