CONFUSIONS BETWEEN DECIMALS, FRACTIONS AND NEGATIVE NUMBERS: A CONSEQUENCE OF THE MIRROR AS A CONCEPTUAL METAPHOR IN THREE DIFFERENT WAYS

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Many students confuse decimal numbers, fractions and negative numbers. Data, some of which is new, is provided to support this observation. Interview data also identifies other confusions between number lines and number-line hybrids and between zero and one. These observations are explained by drawing attention to the use of the mirror as a conceptual metaphor in three different ways for understanding the number system. It underpins the usual positive/negative number line, links natural numbers and their reciprocals and operates in a pseudo number line related to place value columns. Students mentally merge the components that are the images under the analogical mapping of the same mirror feature. This extends recent work on metaphors in mathematics itself to their role in understanding mathematics.

INTRODUCTION

Sometime ago I (the first author) was sitting at the back of a class watching an excellent Year 8 lesson on ratio and scale. The students measured small plastic model animals and had to find the size of the real animal, given the scale factor. The girl alongside me correctly calculated that the length of the pig would be 0.9m, but then seemed puzzled. I asked her to show me how big the real pig would be, expecting her to indicate a length along the desk with her two hands. However, after a long pause, she pointed out the window to the left and said that it would be "a long way, out there". Apparently, she was confusing 0.9 and –9. At the time, I thought that the confusion was surprising but that it was simple to explain. Mathematics lessons must have made so little impact on this girl that she had not learned adequately the meanings of the two symbols: the dot for the decimal point and the dash for the negative. Rather like an English speaker beginning to learn French might confuse an acute or grave accent, she confused the meanings of the dot and the dash. I now reject this simple explanation and propose that this confusion is in fact deep, arising from the use of the mirror as the common conceptual metaphor that underpins comprehension of negative numbers, decimals and fractions and place value. The phrase *conceptual metaphor* is used in the sense of Nunez (2000), who describes this as the "cognitive mechanism by which the abstract is comprehended in terms of the concrete" (p I-6).

In the following sections, we first summarise evidence that confusion between decimal numbers, fractions and negative numbers is common and report some new quantitative data. This establishes that there is a phenomenon needing explanation. Second, we report evidence from interviews conducted by the second author which provide clues to the reasons for the decimal/negative number confusion and display a surprising confusion of zero with one and fundamental problems with number lines. Third, we explain these results by demonstrating how the conceptual metaphor of the mirror is involved in three different ways in understanding the number system and propose that students mentally merging these three "mirrors" explains the observations outlined above. These ideas are explained in more depth and further evidence is presented in Stacey and Helme (submitted). Whereas previous work on conceptual metaphors has concentrated on mathematics as a discipline (e.g. Nunez, 2000), this paper examines conceptual metaphors in students' thinking.

Unless specifically qualified, in this paper the terms *decimal numbers* and *fractions* always refer to mathematically positive numbers. The term *negative number* is used in two ways: in the standard mathematical sense and also to indicate a number "less than zero", which some of our interviewees wish to distinguish from standard negative numbers. Space does not permit a full discussion of this.

CONFUSING DECIMALS, FRACTIONS AND NEGATIVE NUMBERS

Confusions between decimals and fractions

Misconceptions about the meaning of decimal numbers have been documented in many parts of the world and widely studied. The task of comparing the size of two or more decimals (e.g. identifying which of 2.4 and 2.375 is the larger) has been found to be very revealing and has been used as the basis for studies of misconceptions (Resnick et al, 1989; Stacey and Steinle, 1999). One innovation introduced by Stacey and Steinle in developing their decimal comparison test was to include decimals of equal length. It had not been expected that many students would make errors when selecting the larger from a pair of decimals such as 0.3 and 0.4 or 2.64 and 2.57, but a significant number of students made errors on all items of this type on the test. Stacey and Steinle were interested to identify patterns of thinking that significant groups of students were using consistently on all items. They realised that students who were consistently incorrect on comparisons where the longer decimal is larger, consistently correct on comparisons where the shorter decimal is larger and consistently incorrect on equal length comparisons (such as 2.64 / 2.57) may have been interpreting decimals as reciprocals of whole numbers or as other fractions. (Note that this is not the "fraction rule" of Resnick et al, 1989.) For example, students may be identifying 0.3 as something like one third and 2.64 as something like two and one sixty fourth or as two sixty fourths or similar. Evidence that some students think in this way comes from production tasks, as used by Irwin (1996) and others. Table 1 shows how the incidence of this type of thinking varies from Grade 5 to Year 10 (ages about 10 to 15). This is previously unpublished data, based on a test of 24 carefully chosen comparisons from the longitudinal study reported by Stacey and Steinle (1999).

Table 1. *Percentage of students consistently interpreting decimals as reciprocals*

Grade Level	5 (N=963)	6 (N=1465)	7 (N=2297)	8 (N=2102)	9 (N=1645)	10 (N=1066)
Percentage	7.2%	4.8%	5.2%	7.1%	4.3%	3.3%

Confusions between decimals and numbers less than zero

The patterns of responses that arise from interpreting decimals as reciprocals can also arise from interpreting them as negative numbers. In Table 1, it is expected that the students in Grades 5, 6 and 7, who have not met negative numbers at school, would predominantly be confusing decimals with reciprocals. However, older students may have either confusion (and note the increase at Grade 8, when students do a lot of work on negative numbers). In order to explore this possibility, a later decimal comparison test included a group of three direct comparisons of (positive) decimals with zero: the comparison of 0.6 with 0, of 0.22 compared with 0 and 0.00 compared with 0.134. This test was given to 553 teacher education students, at various stages of their training, from four universities in Australia and New Zealand. The results are reported by Stacey et al (in press). In summary, 73 students (13%) made at least one error on the three comparisons with zero and 50 (9%) made either two or three errors. This was markedly higher than the percentages of students making at least one error on the other types of comparison items (generally about 7%). The item most likely to be correct was the comparison 0.00 with 0.134. The presence of the additional digits encouraged or permitted use of a digit-by-digit comparison strategy. As one student said "*It's the decimal point* (i.e. in 0.00 but not in 0) *for some reason makes the zero seem much more like a zero…the fact that there is a one in that tenths position indicates that it* (i.e. 0.134) *is larger… the decimal point has obviously made it easier for me to see*."

Only a handful of students (about 1%) consistently answered all items on the test according to the reciprocal/negative pattern of thinking described above and also made at least two errors on the comparisons with zero. In fact, most of the students making errors on the comparisons with zero tested as expert on the remainder of the test. This behaviour indicated that about 1% of the teacher education students completely identified decimal numbers with negative numbers and about 7% could order non-zero decimals, but thought that some such as 0.6 and 0.22 were less than zero.

The teacher education students were asked to select comparisons which children would find difficult and to explain why. The explanations canvassed two possibilities. The first was that children might think that decimals are negative or less than zero (first two quotes below) and the second explanation (3rd and 4th quotes) is that zero is bigger than a decimal number because it is a whole number.

"0.22 may be mistaken for a negative number below zero."

"Some kids have trouble with the notion of zero. 0.22, though less than one, is greater than zero. [Children] might see it as negative or less."

"Children are taught that the ones column is larger than the tenths column so assume 0 is bigger than a decimal."

"Because '0' is a whole number (to the left of the decimal point) whereas .7 is a decimal number, they may choose 0, as it's not seen as a 'decimal', a smaller number."

Children's difficulties making comparisons of decimals with zero have also been reported elsewhere. Irwin (1996) reported that some 11 to 13 year-old children placed decimal numbers starting with zero (e.g. 0.5, 0.1) below zero on a number line. Irwin concluded that these attempts at ordering were consistent with a system that pivots around zero as equivalent to the decimal point, rather than around one.

Interview results

In order to find explanations for the data summarised above, individual interviews were held as soon as possible after testing with 7 volunteer students from one of the universities, who had made errors in the zero comparison items. Both types of explanations outlined above were given for the incorrect answers. Sometimes a student's answers contained elements of both. Jocelyn explained her wrong answers this way and drew the number line in Figure 1:

" I think I was thinking that zero is equal to one. So I was thinking half of one is less than zero. I was thinking that 0.5, for example, was half of zero, so was thinking it is less than zero. I was visualising a number line with 0.5 on the left hand side of zero".

______/________________/______/______/___/___/___/___/_____/_
0.9 0.8 0.7 0.6 0.5 0 1 2 3

Figure 1: Jocelyn's number line showing that small decimals are less than zero.

She commented in her interview that this line made no logical sense, because she could see that the numbers on the left were approaching one. Jocelyn believed that the decimal point must have acted as a trigger for thinking the number was less than zero:

"When I think of a fraction, decimal points, I always think of .5, and instead of thinking .5 as half of something, it's half of—it has to be less than zero because it's, I don't know why I thought that. I suppose maybe because it's got that point, the decimal point ……in some way it's less than zero because it's got a point there."

Stuart said that he might have been thinking of 0.6 as a negative number. When asked to draw a number line, his first attempt was not a line as such but the numbers 10, 0 and 0.1 set out from left to right. Next he drew a line with zero in the middle but with positive numbers on the left and negative numbers on the right. When he went on to explain his thinking, he also appeared to be confusing zero with one:

> *"I know 0.6 is a portion of one. I may have been thinking along the lines of 0.6 is less than the whole number zero. Is zero a whole number? I don't even know . . . . . I'm looking at whole numbers as being positives and decimals as being negatives . . . . . decimals aren't, they're just fraction amounts."*

Lisa was another student who may have been confusing the number line with the place value columns. When she explained her response to the comparison of 0 with 0.6 she said*, "I was thinking of the number line...I was thinking it was on the right hand side of zero, so going into the negative area for some reason*". She drew a number line left-right inverted, and placed 0.6 between 0 and -1 "*Normally I draw a number line in this direction* (indicates conventional number line) *but just when I was thinking of the decimals then I immediately drew it as a negative in this direction*". Later in the interview Lisa stated that she also thought of zero as a whole number:

> *" I can remember that I was thinking that 'this is a whole number' and that this is a fraction of a whole number. And that a fraction of a whole number must be smaller than zero which is a whole number."*

When the interviewer discussed the idea of zero being a whole number with Anna, she summarised: "*Logically 0.6 has to be part of a whole, part of one, but I guess it's like zero is being turned into one and parts are trying to be made out of nothing really*." In a revealing instant, as she spoke about zero, she made a circle with her hands, which seemed to indicate simultaneously the shape of the symbol for zero and a unit (perhaps the classic pie of introductory fraction teaching) which could be divided into parts.

THREE APPLICATIONS OF THE MIRROR METAPHOR TO NUMBERS

The data above leads us to seek explanations for the facts that some students think decimals (and fractions) are negative numbers or otherwise less than zero and some students confuse zero and one. In addition, some students confuse decimals and reciprocals, but we feel that this is well enough explained as persistence of an undifferentiated primitive idea that decimals represent the fractional parts of numbers. We propose that the confusion and interference arise from the use of the conceptual metaphor of the mirror in three different ways for understanding numbers. Recent developments in cognitive science represent a move away from the traditional role of reasoning as primarily propositional, abstract and disembodied to viewing it as embodied and imaginative. From this perspective, mathematical reasoning entails reasoning with structures that emerge from our bodily experience as we interact with our environment.

Lakoff and Johnson (see, for example, Johnson 1987) demonstrate how reasoning with metaphors is fundamental to human thinking and communication by pointing out how everyday language uses common ideas as metaphors to convey abstract concepts. Nunez (2000) and Lakoff & Nunez (1997) apply these ideas to mathematics demonstrating reasoning through metaphors, such as "numbers are points on a line"; "variables are boxes with numbers inside" or "an equation is a balance". These instances view a less familiar target situation (numbers, equations, variables) through the lens of a familiar, concrete source situation (lines, balances, boxes).

The key feature of metaphor is that one domain is conceptualised in terms of another. We propose that aspects of numbers relevant to the problems outlined above are conceptualised in terms of a mirror. A mirror as a conceptual metaphor has three basic components: the real objects, their images (reflections) and the mirror position (some sort of line of symmetry/balance point/pivot/axis). In order to make a conceptual metaphor, the relations between these basic components must also translate from source to target. With a mirror, each real object has its own clearly identified image. The images share many of the features of the real objects and have the same spatial relation to each other, although with an inversion so that in the image world, things are "the other way around", but otherwise the same. There is, however, a critical asymmetry between the real object and the image: the image is a reflection of the real object, the real object is not a reflection of the image. The real object is primary and the image exists in relation to this, not in its own right. In our more extensive paper (Stacey and Helme, submitted), we link this asymmetry of opposites to the linguistic phenomena of positive and negative terms and marked and unmarked adjectives (Clark and Clark, 1977).

The conceptual metaphor of the mirror with natural numbers functioning as the real objects is used in three different ways in understanding numbers as displayed in Table 2. First, in the classic number line, the positive and negative numbers are balanced around zero. The images are the negative numbers {-1, -2, -3, -4, …} (with later extension to other numbers). In formal mathematical terms, these images are the additive inverses of the natural numbers. Second, in an instance less commonly perceived spatially, the positive numbers and their reciprocals are balanced around the natural number 1. The images are the unit fractions {1/2, 1/3, ¼, …} (again with later extension). In formal mathematical terms, these images are the multiplicative inverses of the natural numbers. We contend that this is also conceptually a mirror, because of the way in which the (unit) fractions are conceptualised in terms of the whole numbers and their basic relations (such as size) are similar but inverted.

The third spatial arrangement with a mirror as conceptual metaphor relates to the number line drawn by Stuart. Here the real objects are not quite the natural numbers but the values of the place value columns {ones, tens, hundreds, thousands, . . . }. Their images are the "fractional" place value columns {tenths, hundredths, thousandths, . . .}. The evidence from the interviews leads us to believe that the spatial arrangement of the usual place value numeration is seen by many of our students as some sort of "number line" along which numbers are distributed. This model, which like the other two stretches out infinitely in both directions, has whole numbers of increasing value on the left side of the decimal point and "decimal numbers" of decreasing value on the right. The mirror position is unclear to many students, who may see it as the decimal point, rather than the ones column.

Table 2. *Comparison of features of three mirror metaphors*.

Aspects of mirror metaphor	**Positive/negative mirror metaphor**	**Reciprocals mirror metaphor**	**Place value mirror metaphor**
Mirror position	0	1	Ones column (not decimal point)
Real objects	Natural numbers, represented by points or positions on the line.	Natural numbers, represented by points or positions on the line.	Values of places {tens, hundreds, etc} or vaguely numbers without decimal part
Images	Negative numbers, represented by points or positions on the line.	Unit fractions, represented by points or positions on the line.	Values of places {tenths, hundredths, etc} or vaguely decimals with zero integer part
Direction of increasing size	Increasing to right (monotonic)	Increasing to right (monotonic)	Increasing to left (but not really monotonic)
Extent of "number line"	From – **infinity** to +**infinity**	From 0 to + **infinity**	From long decimals to long whole numbers

The two confusions above can now be seen as confusions between the different targets of the metaphorical mappings of the same source features. Students who think decimals (and fractions) are negative numbers are merging the different images of natural numbers. They may have merged the images in the first two columns or the first and third. Confusion of 0 and 1 is merging of the mirror positions, and also relates to the decimal point as the significant "divider" (mirror position) between whole numbers and others. Merging the number line and place value "columns", produces a hybrid where the "whole number" part of the place value system is placed on the "positive" side of the number line and the "decimal" part of the place value system on the "negative" side of the number line. As Lisa commented: *"I get my number lines mixed up*".

CONCLUSION

The discussion above outlines an explanation of the confusions between decimals, fractions and negative numbers that are certainly common amongst school students and teacher education students. In summary, the basic elements of the explanation are:

(i) that the natural numbers are the primary elements from which concepts of other numbers are constructed,

(ii) that the metaphor of the mirror is involved in the psychological construction of fractions, negative numbers and place value notation for decimal numbers, although in three different ways,

(iii) that the observed confusions result from students' merging (confusing or not distinguishing between) the different targets of the same feature of the mirror metaphor under the different analogical mappings.

Two of the mirrors are recognised within the formal mathematical system (additive and multiplicative inverses), but the place value mirror and its associated "number line" is only a psychological construct. It is hoped that this example will further the exploration of the role of conceptual metaphors in students' mathematical thinking.

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