New literacies for mathematics: a new view of solving equations.

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Abstract

Mathematical literacy is an individual's capacity to use mathematics as a fully functioning member of a society. Information technology has changed and is still rapidly changing what constitutes mathematical literacy for students in today's schools. This paper describes how mathematical literacy is changing by focussing specifically on one very new aspect: what students will need to know about solving equations when they live in a world equipped with computer software which can perform all the solving routines. Graphical and numerical methods are conceptually simple but can be highly efficient. For algebraic methods, students no longer need a high level of technical skill but the need for fundamental understanding is undiminished.

Mathematical literacy

The phrase "mathematical literacy" is now being used to describe students' capacity to use their mathematical knowledge for informed citizenship. PISA, the new international assessment of 15 year old students that is being conducted under the auspices of the OECD (1999), defines mathematical literacy thus:

> Mathematical literacy is an individual’s capacity to identify and understand the role that mathematics plays in the world, to make well-founded judgements and to engage in mathematics, in ways that meet the needs of that individual’s life as a constructive, concerned and reflective citizen. (OECD PISA, 2001)

The PISA study draws its understanding of mathematical literacy from socio-cultural theories of the structure and use of language. Gee (1996), for example, uses the term *literacy* to refer to the human use of language, the primary function of which is to scaffold the performance of social activities. A literate person knows the resources of the language and can use these for different social functions. In the same way a mathematically literate person knows the resources that mathematics offers and can use these for a variety of purposes in the course of everyday or professional life.

The resources that mathematics offers to a problem solver include facts, concepts and procedures. Concepts provide the way in which a situation is understood and mathematised so that a problem can be crystallised in mathematical terms, and the procedures are needed to solve the mathematical problem. To harness the power of mathematics students need to know facts, concepts and skills, the structure of ideas in the domain and how a situation can be mathematised. New technologies are altering the essential understandings of all of these.

In this paper, we will outline some ideas about what a student who has a high level of mathematical literacy might know about solving equations in the new technological environment. We have selected this topic for two reasons. It is central to the school curriculum and therefore is deserving of attention. In addition, it is affected in interesting ways by software that is now commonplace (function graphing software and spreadsheets) and by other software that is not yet commonplace but may soon be owned by many senior school students (computer algebra systems).

Our interest in this topic is stimulated by our CAS-CAT research project (http://www.edfac.unimelb.edu.au/DSME/CAS-CAT), which is examining the effect on curriculum, pedagogy and assessment of students using calculators (really hand-held computers) that contain a computer algebra system (CAS). These calculators have all the functionality of a scientific calculator, are programmable, can plot graphs to any scale and automatically provide basic information such as intercepts and slopes, have a statistics facility and can deal with lists of numbers like a basic spreadsheet. These features have been available on graphics calculators for about a decade. The new feature is the capacity to deal with symbols algebraically rather than numerically. There are menu items for factorising and expanding, solving and substituting, for example, as well as trigonometric identities and calculus. An illustration of the difference in capability between a graphics calculator and a CAS calculator is given in Table 1. Computer algebra systems (eg Derive, Mathematica, MuMath and Maple) have been available for computers for many years. Their availability on a hand-held machine makes their use feasible in schools.

Table 1 *Sample Differences in Equation Solving with Graphics and CAS Calculators.*

Equation	**Graphics Calculator**	**CAS Calculator**
$3e^{0.5t} = 4$	t = 0.58 can be located as intercept	$t = 2\,ln\frac{4}{3}$ or t = 0.58
$3e^{at} = b$	unable to solve this	$t = \frac{ln\frac{b}{3}}{a}$ or $t = \frac{ln\,b - ln\,3}{a}$

This article is intended to present our current thinking about the direction of long-term changes in curriculum that may occur to achieve mathematical literacy for a new generation of students, focussing on solving equations. First we will outline some changes that we feel are already underway. These arise from the technologies of graphics calculators, spreadsheets and computer graphing programs, that are essentially numerical. Secondly we discuss more speculatively the changes that might arise from adjusting to the impact of automated symbolic algebra.

Solving equations graphically and numerically

In a traditional school curriculum, solving equations algebraically was given the highest priority. This is because there are powerful techniques and important ideas to be learned (and these are discussed below), but also because solving equations by numerical trial and error or by graphing were theoretically possible but in practice cumbersome methods and so not the methods of choice. If an equation could be solved algebraically, then this was clearly best.

In the modern workplace, this is not the case. Spreadsheets (or the table capability of graphics calculators) enable trial and error solutions to equations to be quick and simple. Figure 1 shows part of a table of numbers used to solve the equation $x^2+5x+6=-2x-2$. The columns of values for functions y1 (x^2+5x+6) and y2 ($-2x-2$) can be quickly produced and checked to identify where they are equal. A graphical approach is also simple. Intersection points can be manually or automatically located. Sample output is also given in Figure 1.

Pedagogically, including graphical and numerical approaches in the curriculum gives some students a better understanding of what a solution of an equation means. They can lose sight of this under the weight of learning the algebraic manipulations. The numerical approach through spreadsheets is also conceptually simpler for problem solving in the workplace. A person does not have to conceptualise a problem (e.g. finding a break-even point for a financial venture) as solving an equation or dealing with a function; instead they can simply try the numbers.

What are the new elements of mathematical literacy for solving equations numerically and graphically? Our list includes:

- recognising that there is a range of equation solving methods and that graphical and numerical methods are often a reasonable choice
- knowing techniques for efficiently setting up and searching tables and lists
- being able to choose an appropriate viewing window for a graph

- appreciating that there may be multiple solutions to an equation and knowing approximately where they may be
- not being fooled by the pseudo-accuracy that technology sometimes gives (a solution of 2.5 exactly may be given automatically as 2.499852751839 – twelve decimal places but all wrong!)

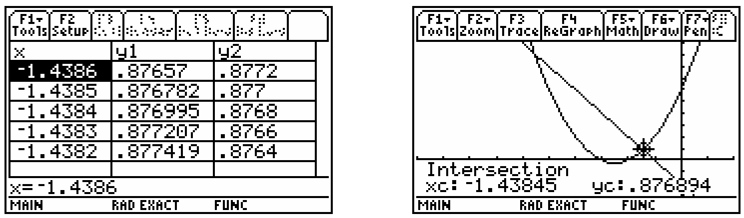
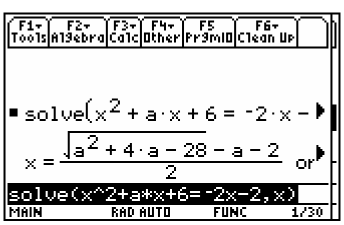


Figure 1: Screen dumps showing numerical and graphical solution for $x^2+5x+6=-2x-2$

Solving equations algebraically

The above new literacies arise from the technology for graphical and numerical approaches for solving equations. We also now have the technology for algebraic approaches. Use of a CAS enables students to find exact solutions to equations, including those containing a parameter. Figure 2 shows the output when a TI 89 calculator is used to solve $x^2+5x+6=-2x-2$ and $x^2+ax+6=-2x-2$. Using the solve feature, exact answers are obtained with the answers given as surds. This is a major difference to a



graphical or numerical approach where only approximate answers are possible. Approximate values can be obtained when required.

Figure 2: Solution for $x^2+5x+6= -2x-2$ and $x^2+ax+6= -2x-2$ using a CAS calculator

Access to a CAS allows for a reconsideration of how we approach solving equations with students. Even with a highly capable automated symbolic system, solving equations is not a routine task and will remain an important part of the school curriculum. The solve feature on a CAS is not always able to solve an equation directly. In this situation, students will require strategies for making the problem amenable to solution. Moreover, we want students to develop a good knowledge of the fundamental principles of solving equations because of their intellectual importance and centrality to mathematics.

What are the new elements of mathematical literacy for solving equations in an era when CAS is available? Our list includes:

- understanding the role of the most basic "equation solving" operations:
  - "do the same to both sides" of an equation
  - using inverse operations
  - using the null factor law
- being sufficiently familiar with algebraic manipulation to be able to modify an equation before input or to recognise calculator output that is in non-standard form
- being able to recognise the basic form of an equation (or set of equations)
- knowing the nature of the solutions of equations of various forms.

Space does not permit us to discuss all aspects of this list. However, we will illustrate the main points with several examples.

One of the key elements in being able to solve equations is the ability to recognise the form of an equation, or set of equations. This is important for solving equations by hand because it gives access to the standard techniques (which are also used by CAS). So, for example, linear equations (in one or more variables) are easy to solve by a standard technique, as are polynomials. Solving an equation such as $e^{2x}-3e^{x}+4=0$ depends on being able to recognise it as a quadratic equation in e^x. When this has been recognised, a method of solution follows as does the expectation of multiple solutions. We feel that in the future, stressing the form of an equation may be a more prominent part of teaching.

Teachers may find it useful, for example, to look at equations such as $2x^2+4=5$, $2x^{0.5}+4=5$ or $2\sin x+4=5$ and observe that they are all of the form $a\,f(x)+b = c$. Students can write $f(x)$ in terms of a, b and c, namely $f(x)=\frac{c-b}{a}$. They could then consider what the solution will be for a range of different functions such as $f(x) = \sqrt{x}$, $f(x) = \cos x$ or $f(x)=3x$. $f(x)$ can be stated for each case using the same result as that used for the general equation. What is important here is the recognition that equations of the form $af(x)+b=c$ can all be rewritten to give $f(x)$. To get to this stage all students will require is an understanding of how to "do the same to both sides" to make $f(x)$ the subject of the equation. After that, the existence (or not) of the inverse of f is important. Recognising form is key to solving equations by hand and it is also key to solving equations with CAS. This is one element in a framework designed by Pierce to describe and organise the algebraic insight required for doing algebra with technology. The complete framework is described in Pierce & Stacey (2001).

Now consider an equation that not all current day CAS can immediately solve. Solving the equation $\frac{pw}{\sqrt{p^2+100}}-1=0$ for p in terms of w was one step in the solution to a question on a year twelve examination (Board of Studies, 1998) in Victoria, Australia. How would having a CAS affect this procedure? Just as in a by-hand solution, it is necessary to find a useful equivalent equation that can be easily solved.

When solving this equation with or without CAS, students need to have some sense of what equations are easy to solve. Here, the obvious first step is rewriting the equation in the form $pw=\sqrt{p^2+100}$. A good first step is very often to write the equation without division (fractions) and this is knowledge that will remain useful. Similarly the next step may be to recognise that an equation of this form will be easier to deal with if both sides are squared to remove non-integer powers, eventually resulting in $p=\pm\sqrt{\frac{100}{w^2-1}}$. The result would need to be interpreted in the context of the original question: a step where the CAS is no help at all. With some CAS, students will almost have to carry out the same process as they would if using a 'by-hand' method. With today's CAS, equation solving is certainly still not trivial and we need to ensure that students have a level of algebraic understanding to enable them to sensibly work towards solutions to problems. Further examples of how CAS may impact on students' problem solving are given in Stacey & Ball (2001) and Stacey, Ball, Asp, McCrae, & Leigh-Lancaster (2000).

Conclusion

The ideas above indicate possible directions for long-term change in the curriculum for equation solving, a very central part of mathematical practice. Planning for mathematical literacy for a new generation of students has to incorporate a recognition of the technologies that are likely to be readily available. Our discussion has shown that this has far-reaching consequences for equation solving. Figure 3 summarises our list of the new literacies associated with solving equations with technology. Solving equations has many routine aspects which can be handed over to the machine, but it is far from a routine process. For mathematical literacy with solving equations, a student needs familiarity with numerical and graphical techniques and also to be able to use the technologies that make these methods practical. For algebraic methods, mathematical literacy involves some practical skills and knowledge of procedures, as well as an appreciation of the nature of the fundamental ideas involved. Solving equations is a routine process only when the equations are identified by form. The CAS can undertake the routine work, although students need to be able to interpret answers – both the syntax and the meaning. Algebraic manipulation to put an equation into a known form is an art – perhaps it is now most important to know the major steps available rather than to practise each step in complicated instances. Finally, a mathematically literate person will understand the major intellectual ideas that are harnessed to solve equations, which has been a central endeavour of mathematics over many centuries.

Aspects of new literacies for using technology to solve equations

- Recognising that numerical, graphical and algebraic methods are available.
- Knowing techniques for efficiently setting up and searching tables and lists.
- Being able to select an appropriate viewing window for a graph.
- Appreciating that there may be multiple solutions to an equation and knowing approximately where they may be.
- Dealing with accuracy and pseudo accuracy of technology.
- Understanding the role of the most basic "equation solving" operations:
  - "do the same to both sides" of an equation
  - using inverse operations
  - using the null factor law.
- Being sufficiently familiar with algebraic manipulation to be able to modify an equation before input or recognise calculator output that is in non-standard form.
- Being able to recognise the basic form of an equation (or set of equations.)
- Knowing the nature of the solutions of equations of various forms.

Figure 3: Aspects of literacies for solving equations with technology

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