



Year 11

Exponential

Functions

CAS-CAT project



Exponential Functions – A Unit Plan

Unit Notes

1. A starter context or problem posed to motivate study of exponential functions – what are the salient features of an ‘exponential’ relationship? The ‘Grains of Rice Problem’ illustrates these features well (Worksheet 1). It allows the students to appreciate the notion of a constant ratio and rates of change.

2. In this unit, some key questions include:

- How are exponential functions similar/different to polynomial functions?
- What characterise an exponential relationship (eg constant successor ratio)?
- What is the effect of changing the base value in an exponential function?
- What relationships do you know that involve a constant successor ratio?
- What is the domain and range of exponential and logarithmic functions, and why these values?

2. Revisiting the Index Laws and introducing the log laws is necessary for the student to make sense of exact results such as **solve $(2^x=3,x) = \log 3/\log 2$** and **solve $(2^x=-1,x) = \text{false}$**

3. It is appropriate to focus on restating of index equations and expressions in log format and vice versa, so that students become versatile in ‘rewriting’ and making sense of the different forms of equivalent expressions.

4. The inclusion of ‘Transformation Creations’ is to present an alternative approach to the more usual direct instruction about parameters affecting graph shape and location. Students are encouraged to ‘discover’ how each parameter influences the above.

5. ‘Scale Issues Worksheets’ are to confront possible misconceptions and incomplete understanding of the role of the viewing window dimensions in representation of a function’s graph.

Mathematical Methods (CAS) Unit 2: Exponential Functions

Study Design for MM (CAS) Unit 2	Textbook Examples/Worksheets	Technology Ideas etc
<ul style="list-style-type: none"> •Recognition of equivalent expressions and simplification of algebraic expressions involving functions and relations, including use of exponent and logarithm laws •apply index (exponent) laws to manipulate and simplify expressions involving these terms by hand in simple cases; 	<p>Starter Worksheet 1 – Grains of Rice Other Possible Starters – similar stimulus question about doubling money, paper folding etc</p> <p>MQ Ex 4A, 4B (Index Laws) ESS Ex 13B (Index Laws)</p> <p>MQ Ex 4C (Indicial Equations) ESS Ex 13C,D (Indicial Equations)</p>	<p>Students will need to be able to do simple cases by hand. The CAS calculator can be used to reinforce the links between index laws and log laws.</p> <p>Link to graphical meaning of ‘solving’</p>
<ul style="list-style-type: none"> •Graphs of $y = Aa^{kx} + B$, where $a \in \mathbb{R}^+$, $a \neq 1$, for simple cases of A, k and $B \in \mathbb{R}$; •use of parameters to represent a family of functions and general solutions of equations involving these functions; •The graph of $y = \log_a x$ as the inverse of $y = a^x$, for simple cases of a where $a \in \mathbb{R}^+$, $a \neq 1$, informal discussion of the inverse relationship; •Use of inverse functions to solve equations 	<p>Worksheet 2 - Graphs of $f(x) = a^x$ Worksheet 3 - Transformation Designs</p> <p>MQ Ex 4D (Exponential Graphs) ESS Ex 13A (Exponential Graphs) Worksheet 4 – Scale Issues</p> <p>MQ Ex 4E (Logarithms) ESS Ex 13E (Logarithms) ESS Ex 13G (Logarithm Graphs)</p> <p>Worksheet 5 – Verifying the Log Laws</p> <p>MQ Ex 4F (Solving Log Equations) ESS Ex 13F (Solving Log Equations)</p>	<p>Use of parameters, comparison with parameters for other function families</p> <p>Many exp. fns change rapidly – students need to be versatile in working with the viewing window</p> <p>Use of Solve Function Output requires students understand logs as inverse of exponentiation</p> <p>Worksheet 4 helps students make sense of the logarithmic function</p>
<ul style="list-style-type: none"> •Illustration of the application of exponential functions of the form $y = Aa^{kx} + B$, for simple cases of A, k and $B \in \mathbb{R}$, in data modelling such as growth and decay models in populations and the physical world, appreciation and depreciation of value in finance and the interpretation of initial value, rate of growth or decay and long run value in these contexts and their relationship to the parameters A, k and B. 	<p>Worksheet 6 – Population Change</p> <p>MQ Ex 4G (Applications) ESS Applications</p> <p>Worksheet 7 – Euler’s Number</p>	<p>Emphasise use of function notation for substitution eg $y_1(3)$ and Solve ($y_1(x)=7,x$)</p> <p>Use of modelling based on collected data (eg rebound height) as well as functional models</p>

Evans, M., Lipson, K., Wallace, D., McCoy, T., & Blane, D. (1999). *Essential Mathematical Methods 1&2*.

Melbourne: Cambridge University Press.

Nolan, J., Phillips, G., Allen, R., & Phillips, D. (2000). *Maths Quest: Mathematical Methods 11: Mathematical Methods VCE mathematics units 1 and 2*. Queensland: John Wiley & Sons Australia Ltd.

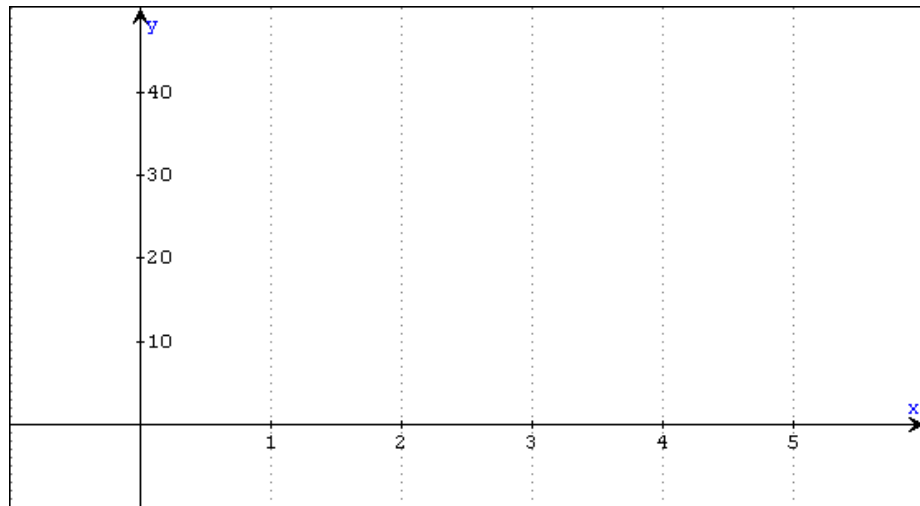
<http://www.vcaa.vic.edu.au/vce/studies/MATHS/caspilot.htm> - CAS pilot, Victorian Curriculum and Assessment Authority, Victoria.

Worksheet 1 – Grains of Rice

A traditional problem in Mathematics concerns the number of grains of rice or wheat which could be put on a chess board if there were one on the first square, two on the second, four on the third and so on with the number being doubled for each successive square.

- (i) Fill in the table and plot the corresponding graph for $x = 1, 2, \dots, 6$

Number of square	1	2	3	4	5	6
Number of grains						



- (ii) Join the points in the most appropriate manner.
 (iii) Explain whether or not you think it is reasonable to join the points.
- (iv) How many grains would be on the sixty fourth square?
- (v) Formulate a rule that calculates the number of grains on the x^{th} chessboard square
- (vii) Plot this rule on your calculator and view it in a suitable window (think hard!).
 (viii) Draw an appropriately scaled sketch of this in the space provided

Worksheet 2 – Exploring Graphs of $f(x) = a^x$

In the following questions, you will be exploring how the ‘base’ a in an exponential function impacts the behaviour of the function for various given x values.

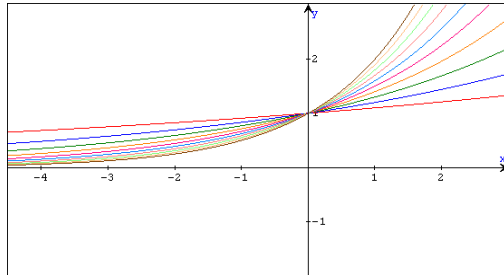
1.
 - (i) Make a table showing the values of 3^x for
 - (ii) Use your graphics calculator to generate the graph of for
 - (iii) Trace along your graph and estimate the values of when x is -1.5, -0.2, 1.5, 1.8, 2.2 and 2.5.
 - (iv) Use your answers in part (iii) to estimate the values of $3^{1.5} \times 3^{2.5}$, $3^{-1.5} \times 3^{2.5}$, $3^{-0.2} \times 3^{2.2}$, $3^{1.8} \div 3^{-0.2}$ and $3^{1.5} \div 3^{2.5}$
 - (v) Compare your estimates in (iv) with values in the table and write conjectures in English and Mathematical sentences to summarize your comparisons.
 - (vi) Test your conjectures with other examples from your graph and table.

2.
 - (i) Make a table showing the values of for
 - (ii) Use your graphics calculator to generate the graph of for
 - (iii) Trace along your graph and estimate the values of when x is -1.5, -0.2, 1.5, 1.8, 2.2 and 2.5.
 - (iv) Use your answers in part (iii) to estimate the values of $0.5^{1.5} \times 0.5^{2.5}$, $0.5^{2.2} \times 0.5^{-0.2}$, $0.5^{2.5} \div 0.5^{1.5}$, $0.5^{1.8} \div 0.5^{-0.2}$ and $0.5^{1.5} \div 0.5^{2.5}$
 - (v) Compare you estimates in (iv) with values in the table and write conjectures in English and Mathematical sentences to summarize your comparisons.
 - (vi) Test your conjectures with other examples from your graph and table.

Worksheet 3 – Transformation Designs

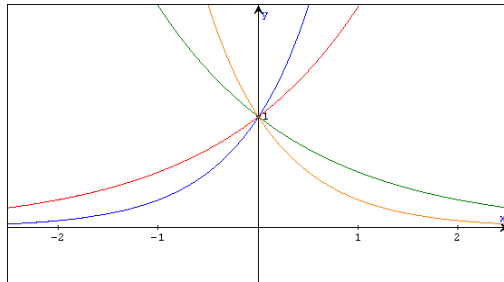
Use exponential functions of the form $f(x) = a \cdot b^{x-h} + k$ to recreate the designs below in the windows given. Record the function you have used in the space provided.

Twisted Ribbon



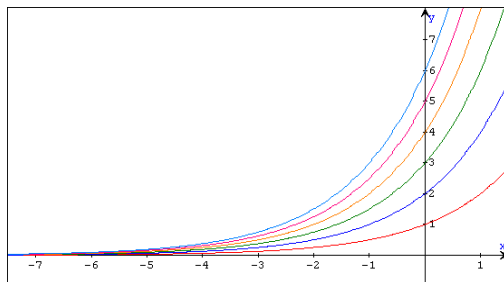
Record functions used

Bow



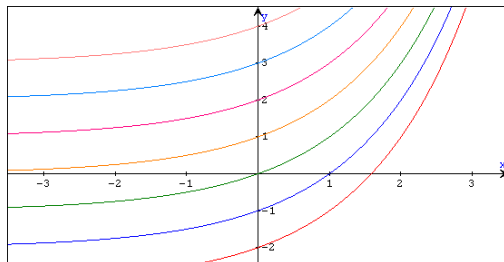
Record functions used

Steep Slide



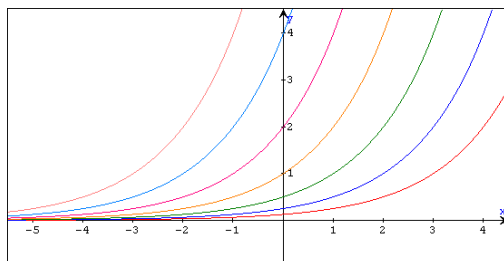
Record functions used

Highway Bend



Record functions used

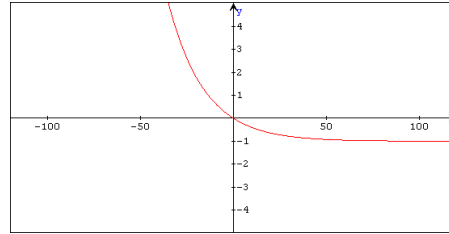
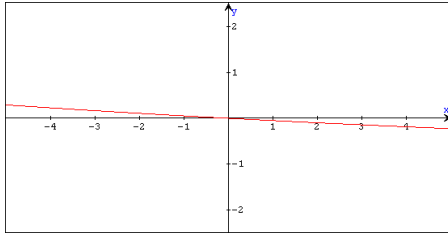
Deck Chair



Record functions used

Worksheet 4 – Scale Issues

Graphs of functions don't always appear in the way we expect, and this is often related to the boundaries of the viewing window, usually denoted as [Xmin, Xmax] by [Ymin, Ymax]



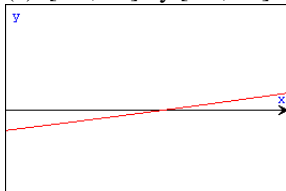
For the graph of each function listed below

- Enter the rules and plot them in the default viewing window
- Zoom or adjust window settings as necessary to find the 'big picture'
- Find any intercepts with the axes, and the location of any asymptotes
- Draw a quick sketch of the graph, clearly labelling and scaling each axis

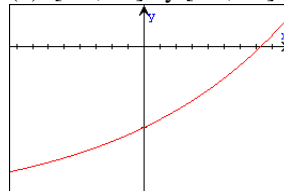
1. $f(x) = 0.95^{x-1}$	2. $f(x) = 0.9^{2-x}$
3. $f(x) = 2^x + 10$	4. $f(x) = 25 - 0.2(1.5)^x$

5. The graph of $f(x) = 1.1^x - 2$ can be made to look like the following graphs. In each case, tried to reproduce the same graph and then record the window values that might create that view.

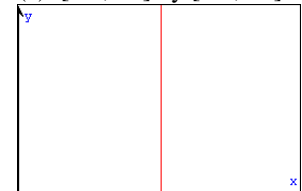
(a) [,] by [,]



(b) [,] by [,]

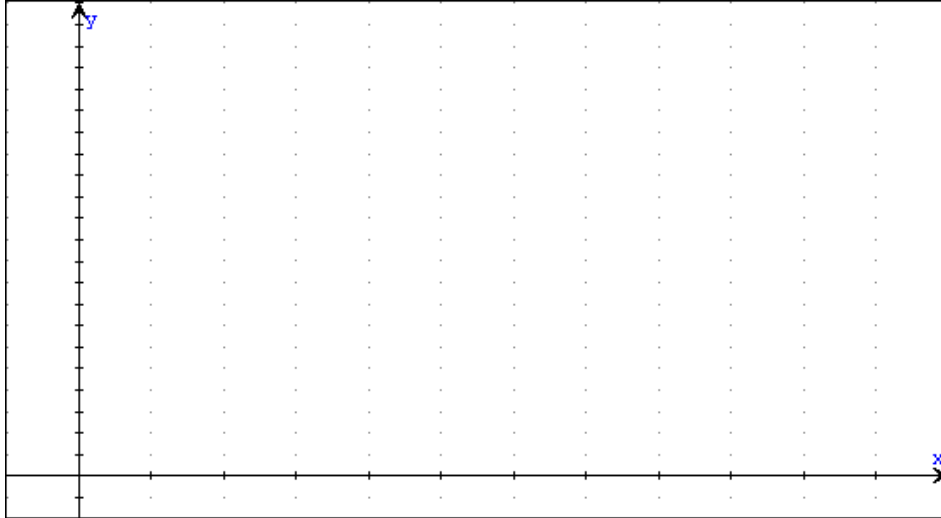


(c) [,] by [,]



Worksheet 5 – Verifying the Log Laws

Use your graphics calculator to plot the graph of $y = \log_{10} x$ for values $x = 1, 2, 3, 4, 5, 6, 10, 20, 30, 40, 50, 60, 120$. Draw a quick sketch of it on the axes provided below, marking the scale on the axes appropriately.



- (i) From your graph (on your calculator) determine the values (rounded to two decimal places) of the logarithms of 1, 2, 3, 4, 5, 6, 10, 20, 30, 40, 50, 60 and 120 and comment.

x	1	2	3	4	5	6	10	20	30	40	50	60	120
$f(x) = \log_{10} x$													

- (ii) Determine the value of $\log 2 + \log 5$ and determine the value of a if $\log a = \log 2 + \log 5$
- (iii) Determine the value of $\log 4 + \log 5$, and determine the value of a if $\log a = \log 4 + \log 5$
- (iv) Determine the value of $\log 60 - \log 3$ and determine the value of a if $\log a = \log 60 - \log 3$
- (v) Determine the value of $\log 120 - \log 6$ and determine the value of a if $\log a = \log 120 - \log 6$
- (vi) Check your results by entering them into the calculator directly
- (v) Comment on your results and explain in your own words

Worksheet 7 – Introducing Euler’s Number

In this task, we explore derivatives of exponential functions, that is, functions of the type $f(x) = b^x$, $b > 0$ and x any real number. It also introduces a special number, e .

1. In Y1, define the function $f(x) = 5^x$.

(a) Calculate $[f(2.001) - f(2)]/0.001$. This is an estimate of the derivative of 5^x at $x = 2$, that is, $f'(2)$.

Calculate $[[f(2.001) - f(2)]/0.001] / f(2)$. This is an estimate of $f'(2)/f(2)$.

(b) Use the same method as (a) to estimate $f'(1)/f(1)$.

(c) Use the same method as (a) to estimate $f'(-2)/f(-2)$.

(d) The results of (a) – (c) above suggest something about the ratio $f'(x)/f(x)$. Write in words what you believe is true.

2. We will now use the graphics calculator to check the results of question 1.

(a) Enter the functions $Y1 = f(x) = 5^x$ and $Y2 = f'(x)$ and plot them. What do you notice about the general shape of the function and its derivative?

(b) Now enter the function $Y3 = Y2/Y1$. By tracing along the graph of $Y3$, check the result of question 1(d).

3. In Y1, change the rule to $f(x) = b^x$. Repeat question 2 for the function $f(x) = b^x$ for the values of b in the following table, and complete the table:

b	$f(x)$	$f'(x)$	$f'(x)/f(x)$
0.5			
1.0			
1.5			
2.0			
2.5			
3.0			

4. Use the results of question 3 to help you find a base b for which the ratio $f'(x)/f(x) = 1$. Try to find this number correct to two decimal places.

5. The number that you have found in question 4 is Euler’s number denoted by ‘ e ’. One reason it is so important is that if we let $b = e$, the Euler number, then $f(x) = b^x = e^x$ has derivative $f'(x) = f(x)$ [because $f'(x)/f(x) = 1$]. Verify this by repeating question 2 for the function $f(x) = e^x$.